FUNCTIONAL
DIFFERENTIAL
EQUATIONS

VOLUME 16
2009, NO 1
PP. 53–72

GLOBAL EXPONENTIAL PERIODICITY FOR THE DISCRETE ANALOGUE OF AN IMPULSIVE HOPFIELD NEURAL NETWORK WITH FINITE DISTRIBUTED DELAYS

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Abstract. The discrete counterpart of a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays is introduced. The continuation theorem of coincidence degree theory is used to obtain a sufficient condition for the existence of a periodic solution of the discrete system considered. By introducing an appropriate Lyapunov functional a sufficient condition is obtained for the uniqueness and global exponential stability of the periodic solution.

Key Words. Discrete Hopfield neural networks, finite distributed delays, periodic impulses, global exponential periodicity.

AMS(MOS) subject classification. 34A37, 39A11

1. Introduction.

A neural network is a network that performs computational tasks such as associative memory, pattern recognition, optimization, model identification, signal processing, etc. on a given pattern via interaction between a number of interconnected units characterized by simple functions. From the mathematical point of view, an artificial neural network corresponds to a nonlinear transformation of some inputs into certain outputs. Many types of neural networks have been proposed and studied in

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the literature and the Hopfield-type network has become an important one due to its potential for applications in various fields of daily life. The model proposed by Hopfield, also known as Hopfield's graded response neural network, is based on an analogue circuit consisting of capacitors, resistors and amplifiers.

Hopfield neural networks have found applications in a broad range of disciplines [8, 9, 10] and have been studied both in the continuous and discrete time cases by many researchers. Most neural networks can be classified as either continuous or discrete. In spite of this broad classification, there are many real world systems and natural processes that behave in a piecewise continuous style interlaced with instantaneous and abrupt changes (impulses). Periodic dynamics of the Hopfield neural networks is one of the realistic and attractive modellings for the researchers. Signal transmission between the neurons causes time delays. Therefore the dynamics of Hopfield neural networks with discrete or distributed delays has a fundamental concern.

In the present paper, we introduce the discrete counterpart of a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays. We apply the continuation theorem of coincidence degree theory to obtain a sufficient condition for the existence of a periodic solution of the discrete system considered. By introducing an appropriate Lyapunov functional we derive a sufficient condition for the uniqueness and global exponential stability of the periodic solution. For works proving the existence of a periodic solution of differential and difference equations by the coincidence degree theory the reader can see [3, 4, 5, 6, 12, 13, 14]. In particular, in [14] the existence of a periodic solution of Hopfield-type neural network with impulses is proved. In [18] one proves the existence of a periodic solution of a discrete-time analogue of a bidirectional associative memory (BAM) neural network with periodic coefficients and finite distributed delays without impulses.

2. Statement of the problem. Main results. We consider a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays, which are formulated in the form of a system of impulsive delay differential equations

$$\frac{dx_i}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j\left(\int_0^\omega g_{ij}(s)x_j(t-s)\,ds\right) + I_i(t), \tag{1}$$
$$t \neq t_k,$$

$$\begin{aligned} \Delta x_i(t_k) &\equiv x_i(t_k+0) - x_i(t_k) \\ &= -\gamma_{ik}x_i(t_k) + \sum_{j=1}^{m} B_{ijk}\Phi_j\left(\int_0^\omega c_{ij}(s)x_j(t_k-s)\,ds\right) + \alpha_{ik}, \end{aligned} \tag{2}$$
$$i = \overline{1,m}, \quad k \in \mathbb{Z},$$

where m is the number of neurons in the network, $x_i(t)$ is the state of the i-th neuron at time t, $a_i(t) > 0$ is the rate at which the i-th neuron resets its state when isolated from the system, $b_{ij}(t)$ is the synaptic connection weight from the j-th neuron to the i-th one, $f_j(\cdot)$ are signal transmission functions of the j-th neuron, ω is the maximum transmission delay from one neuron to another, $g_{ij}(\cdot)$ and $c_{ij}(\cdot)$ are nonnegative delay kernels, $I_i(t)$ is the external input to the i-th neuron, t_k $(k \in \mathbb{Z})$ are the instants of impulse effect which form a strictly increasing sequence, γ_{ik} $(i = \overline{1,m},\ k \in \mathbb{Z})$ are positive constants.

We assume that the above system (1), (2) satisfies the following periodicity conditions: $a_i(t)$, $b_{ij}(t)$, $I_i(t)$ are ω-periodic in t; $t_{k+p} = t_k + \omega$, $\gamma_{i,k+p} = \gamma_{ik}$, $B_{ij,k+p} = B_{ijk}$, $\alpha_{i,k+p} = \alpha_{ik}$. Without loss of generality we can assume that

$$0 < t_1 < t_2 < \cdots < t_p < \omega.$$

The Hopfield neural network (1) is similar to the bidirectional associative memory neural network considered in [18].

We can consider the system (1) for $t > 0$, the impulse conditions (2) for $k > 0$, with initial conditions

$$x_i(s) = \phi_i(s) \quad \text{for} \quad s \in [-\omega, 0], \qquad i = \overline{1,m}, \tag{3}$$

where the initial functions $\phi_i(s)$, $i = \overline{1,m}$, are piecewise continuous with points of discontinuity of the first kind at $t_{-p+1}, t_{-p+2}, \dots, t_{-1}, t_0$. To find an ω-periodic solution of system (1), (2) means to determine the initial functions $\phi_i(s)$ so that the solution of the initial value problem (1), (2), (3) is ω-periodic.

Combining some ideas of [16, 1, 18] we shall formulate the discrete counterpart of system (1), (2). For a positive integer N we choose the discretization step $h = \omega/N$. For the moment we assume N so large that

$$h < \min_{k=\overline{1,p}} (t_{k+1} - t_k).$$

Then each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience we denote $n = [t/h]$, the greatest integer in t/h, and $n_k = [t_k/h]$. Clearly, we will have $n_{k+p} = n_k + N$ for all $k \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$, $n \neq n_k$. This means that the interval $[nh, (n+1)h]$ contains no instant of impulse effect t_k.

We approximate the integral term in (1) by a sum:

$$\int_0^\omega g_{ij}(s) x_j(t-s)\, ds \approx \sum_{\ell=1}^{N} g_{ij}(\ell h) x_j\left((n-\ell)h\right) \varphi(h),$$

where $\varphi(h) = h + O(h^2)$.

Next we approximate the differential equation (1) on the interval $[nh, (n+1)h]$ by

$$\frac{dx_i}{dt} + a_i(nh) x_i(t) = I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell h) x_j\left((n-\ell)h\right) \varphi(h) \right).$$

We multiply both sides of this equation by $\exp\left(a_i(nh)t\right)$ and integrate over the interval $[nh, (n+1)h]$. Thus we obtain

$$(4) \quad x_i\left((n+1)h\right) - x_i(nh) = -\left(1 - e^{-a_i(nh)h}\right) x_i(nh)$$
$$+ \quad \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} \left\{ I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell h) x_j\left((n-\ell)h\right) \varphi(h) \right) \right\}.$$

Henceforth by abuse of notation we write $x_i(n) = x_i(nh)$ and define $\Delta x_i(n) = x_i(n+1) - x_i(n)$ ($i = \overline{1,m}$, $n \in \mathbb{Z}$). For convenience we adopt the notations:

$$\begin{aligned}
A_i(n) &= 1 - e^{-a_i(nh)h} \qquad (i = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}),\\
I_i(n) &= \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} I_i(nh) \qquad (i = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}),\\
b_{ij}(n) &= \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} b_{ij}(nh) \qquad (i,j = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}),\\
g_{ij}(\ell) &= g_{ij}(\ell h)\varphi(h) \qquad (i,j = \overline{1,m},\ \ell = \overline{1,N}).
\end{aligned}$$

Clearly, we have $0 < A_i(n) < 1$. In particular, if $a_i(t) < \frac{1}{\omega}$, then $A_i(n) < \frac{1}{N}$.

With the above notation equation (4) takes the form

$$\text{(5)}\ \Delta x_i(n) = -A_i(n)x_i(n) + I_i(n) + \sum_{j=1}^{m} b_{ij}(n) f_j\left(\sum_{\ell=1}^{N} g_{ij}(\ell)x_j(n-\ell)\right), \quad i = \overline{1,m}, \quad n \neq n_k.$$

Next, for $n = n_k$ the interval $[nh, (n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition (2) by

$$\text{(6)}\ \Delta x_i(n_k) = -\gamma_{ik}x_i(n_k) + \alpha_{ik} + \sum_{j=1}^{m} B_{ijk}\Phi_j\left(\sum_{\ell=1}^{N} c_{ij}(\ell)x_j(n_k-\ell)\right), \quad i = \overline{1,m}, \quad k \in \mathbb{Z},$$

where

$$c_{ij}(\ell) = c_{ij}(\ell h)\varphi(h) \qquad (i,j = \overline{1,m}, \quad \ell = \overline{1,N}).$$

For uniformity of notation we define

$$A_i(n_k) = \gamma_{ik}, \quad I_i(n_k) = \alpha_{ik} \qquad (i = \overline{1,m},\ k \in \mathbb{Z}).$$

Now the difference system (5), (6) can be written in operator form as

$$\text{(7)} \qquad \Delta x = Hx,$$

where

$$(Hx)_i(n) = -A_i(n)x_i(n) + I_i(n) \tag{8}$$
$$+ \begin{cases} \sum\limits_{j=1}^{m} b_{ij}(n) f_j\left(\sum\limits_{\ell=1}^{N} g_{ij}(\ell)x_j(n-\ell)\right), & n \neq n_k, \\ \sum\limits_{j=1}^{m} B_{ijk}\Phi_j\left(\sum\limits_{\ell=1}^{N} c_{ij}(\ell)x_j(n_k-\ell)\right), & n = n_k. \end{cases}$$

We can consider the system (7) for $n \geq 0$, with initial conditions

$$x_i(\ell) = \phi_i(\ell) \quad \text{for} \quad \ell = 0, -1, \dots, -N, \qquad i = \overline{1, m}, \tag{9}$$

where $\phi(\ell) = (\phi_1(\ell), \phi_2(\ell), \dots, \phi_m(\ell))^T$, $\ell = 0, -1, \dots, -N$, are given initial vectors. To find an N-periodic solution of system (7) means to determine the initial vectors $\phi(\ell)$ so that the solution of the initial value problem (7), (9) is N-periodic.

In order to formulate our assumptions, we need some more notation:

$$I_N = \{0, 1, \dots, N-1\},$$

$$\underline{A}_i = \min_{n \in I_N} A_i(n), \quad \overline{A}_i = \sum_{n=0}^{N-1} A_i(n), \qquad i = \overline{1, m}.$$

Now we introduce the following conditions:

H1. $A_i(n+N) = A_i(n)$, $I_i(n+N) = I_i(n)$ for $i = \overline{1, m}$, $n \in \mathbb{Z}$; $n_k \in \mathbb{Z}$ for all $k \in \mathbb{Z}$ and $n_{k+p} = n_k + N$; $b_{ij}(n+N) = b_{ij}(n)$ $(n \neq n_k)$, $B_{ij,k+p} = B_{ijk}$ $(k \in \mathbb{Z})$ for $i, j = \overline{1, m}$.

H2. $\underline{A}_i > 0$, $\overline{A}_i < 1$ for $i = \overline{1, m}$.

H3. The functions $f_j(\cdot)$, $\Phi_j(\cdot)$ $(j = \overline{1, m})$ are Lipschitz continuous on $\mathbb{R}$, that is, there exist positive constants M_j and L_j such that

$$|f_j(x) - f_j(y)| \leq M_j|x - y|, \quad |\Phi_j(x) - \Phi_j(y)| \leq L_j|x - y|$$

for all $x, y \in \mathbb{R}$.

H4. $g_{ij}(\ell) \geq 0$, $c_{ij}(\ell) \geq 0$ for $i, j = \overline{1, m}$, $\ell = \overline{1, N}$.

We again introduce some notation:

$$\overline{I}_i = \max_{n \in I_N} |I_i(n)|, \qquad i = \overline{1, m},$$

$$\overline{b}_{ij} = \sup_{n \neq n_k} |b_{ij}(n)|, \quad \overline{B}_{ij} = \max_{k = \overline{1, p}} |B_{ijk}|, \qquad i, j = \overline{1, m}.$$

For an N-periodic sequence $v(n)$ we denote $\tilde{v} = \frac{1}{N}\sum_{n=0}^{N-1} v(n)$; for $i = \overline{1,m}$

$$\rho_i' = \overline{I}_i + \sum_{j=1}^{m} \overline{b}_{ij}|f_j(0)|, \qquad \rho_i'' = \overline{I}_i + \sum_{j=1}^{m} \overline{B}_{ij}|\Phi_j(0)|, \tag{10}$$

$$\rho_i = [(N-p)\rho_i' + p\rho_i'']/N = \overline{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left[(N-p)\overline{b}_{ij}|f_j(0)| + p\overline{B}_{ij}|\Phi_j(0)|\right].$$

Next we denote

$$\mathcal{M}_j = \max\{L_j,\ M_j\}, \quad j = \overline{1,m},$$
$$G_{ij} = \sum_{\ell=1}^{N} g_{ij}(\ell), \quad C_{ij} = \sum_{\ell=1}^{N} c_{ij}(\ell), \qquad i,j = \overline{1,m},$$
$$\mathcal{B}_{ij} = \max\{\overline{b}_{ij},\ \overline{B}_{ij}\}, \quad \mathcal{G}_{ij} = \max\{G_{ij},\ C_{ij}\}, \qquad i,j = \overline{1,m}.$$

We introduce the $m \times m$ matrices

$$A = \operatorname{diag}\left(\frac{\underline{A}_i\,(1-\overline{A}_i)}{1+N\underline{A}_i},\ i = \overline{1,m}\right), \qquad B = (\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij})_{i,j=1}^{m}. \tag{11}$$

Next we introduce the conditions

H5. $\min\limits_{i=\overline{1,m}}\left(\tilde{A}_i - \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}\right) > 0.$

H6. $\underline{A}_i > \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}$ for $i = \overline{1,m}$.

H7. The matrix $A - B$ is an M-matrix (see [2, 11]).

Clearly, condition **H6** implies **H5** but the converse is not true. Condition **H7** means that the matrix $A - B$ is nonsingular and its inverse has positive entries only.

Now we can state our main results as two theorems which will be proved in the next two sections.

Theorem 1. *Suppose that conditions* **H1–H5**, **H7** *hold. Then the equation* (7) *has at least one N-periodic solution.*

Theorem 2. *Suppose that conditions* **H1–H4**, **H6**, **H7** *hold. Then the N-periodic solution of* (7) *is unique and globally exponentially stable.*

3. Proof of the existence of a periodic solution. We shall prove Theorem 1 using Mawhin's continuation theorem [7, p. 40]. To state this theorem we need some preliminaries:

Let $\mathbb{X}, \mathbb{Y}$ be real Banach spaces, $L : \operatorname{Dom} L \subset \mathbb{X} \longrightarrow \mathbb{Y}$ be a linear mapping, and $H : \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \longrightarrow \mathbb{X}$ and $Q : \mathbb{Y} \longrightarrow \mathbb{Y}$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, then the mapping $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)\mathbb{X} \longrightarrow \operatorname{Im} L$ is invertible. We denote the inverse of this mapping by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping H will be called L-compact on $\overline{\Omega}$ if $QH(\overline{\Omega})$ is bounded and $K_P(I - Q)H : \overline{\Omega} \longrightarrow \mathbb{X}$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \longrightarrow \operatorname{Ker} L$.

Now Mawhin's continuation theorem can be stated as follows.

Lemma 1. *Let L be a Fredholm mapping of index zero, let $\Omega \subset \mathbb{X}$ be an open bounded set and let $H : \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(*a*) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Hx$;*

(*b*) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QHx \neq 0$;*

(*c*) $\deg(JQH, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where* $\deg(\cdot)$ *is the Brouwer degree.*

Then the equation $Lx = Hx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

It is much easier to apply this lemma to difference equations than to differential equations since in the former case all spaces are finite dimensional.

Before we proceed further, we shall recall the definition of Brouwer degree [15].

Suppose that M and N are two oriented differentiable manifolds of dimension n (without boundary) with M compact and N connected and suppose that $f : M \rightarrow N$ is a differentiable mapping. Let $Df(x)$ denote the differential mapping at the point $x \in M$, that is the linear mapping $Df(x) : T_x(M) \rightarrow T_{f(x)}(N)$. Let $\operatorname{sign} Df(x)$ denote the sign of the determinant of $Df(x)$. That is the sign is positive if f preserves orientation and negative if f reverses orientation.

Definition 1. Let $y \in N$ be a regular value, then we define the *Brouwer degree* (or just *degree*) of f by

$$\deg f := \sum_{x \in f^{-1}(y)} \operatorname{sign} Df(x).$$

It can be shown that the degree does not depend on the regular value y that we pick so that $\deg f$ is well defined.

Note that this degree coincides with the degree as defined for maps of spheres.

Let us choose $\mathbb{X} = \mathbb{Y} = \{x(n) = (x_1(n), x_2(n), \dots, x_m(n))^T : x(n+N) = x(n),\ n \in \mathbb{Z}\}$. If we define $|x_i| = \max_{n \in I_N} |x_i(n)|$, $\|x\| = \sum_{i=1}^{m} |x_i|$, then $\mathbb{X}$ is a Banach space with the norm $\|\cdot\|$. For $x \in \mathbb{X}$, let Hx be defined by (8), $Lx = \Delta x$ and

$$Px = Qx = \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m)^T.$$

Then $\operatorname{Ker} L = \{x \in \mathbb{X} : x = h \in \mathbb{R}^m\}$ (vectors with components independent of n), $\operatorname{Im} L = \{x \in \mathbb{X} : \sum_{n=0}^{N-1} x_i(n) = 0,\ i = \overline{1,m}\}$ is a closed set in $\mathbb{X}$, and $\operatorname{codim} L = m$. Thus L is a Fredholm mapping of index zero. It is easy to see that P and Q are continuous projectors and $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, and H is L-compact on $\bar{\Omega}$ for any bounded set $\Omega \subset \mathbb{X}$. Moreover, in condition (c) of Lemma 1 the isomorphism J can be taken as the identity operator I.

Now we will derive some estimates for the solutions x of the operator equation $Lx = \lambda Hx$ for $\lambda \in (0, 1)$, that is,

$$\Delta x_i(n) = \lambda (Hx)_i(n), \quad n \in I_N,\ i = \overline{1,m}. \tag{12}$$

First from (12) and (8) for $n \neq n_k$ we obtain

$$|\Delta x_i(n)| \le A_i(n)|x_i(n)| + |I_i(n)| + \left| \sum_{j=1}^{m} b_{ij}(n) f_j\left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell) \right) \right|$$

$$\le A_i(n)|x_i| + \overline{I}_i + \sum_{j=1}^{m} \overline{b}_{ij} M_j \sum_{\ell=1}^{N} g_{ij}(\ell)|x_j(n-\ell)| + \sum_{j=1}^{m} \overline{b}_{ij}|f_j(0)|$$

$$\le A_i(n)|x_i| + \rho_i' + \sum_{j=1}^{m} \overline{b}_{ij} M_j G_{ij} |x_j|$$

$$\le A_i(n)|x_i| + \rho_i' + \sum_{j=1}^{m} \mathcal{B}_{ij} \mathcal{M}_j \mathcal{G}_{ij} |x_j|,$$

where ρ_i' was introduced in (10).

Similarly, for $n = n_k$ we have

$$|\Delta x_i(n_k)| \le A_i(n_k)|x_i(n_k)| + |I_i(n_k)| + \left| \sum_{j=1}^{m} B_{ijk}(n)\Phi_j \left(\sum_{\ell=1}^{N} c_{ij}(\ell) x_j(n_k - \ell) \right) \right|$$

$$\le A_i(n_k)|x_i| + \overline{I}_i + \sum_{j=1}^{m} \overline{B}_{ij} L_j \sum_{\ell=1}^{N} c_{ij}(\ell)|x_j(n_k - \ell)| + \sum_{j=1}^{m} \overline{B}_{ij}|\Phi_j(0)|$$

$$\le A_i(n_k)|x_i| + \rho_i'' + \sum_{j=1}^{m} \overline{B}_{ij} L_j C_{ij} |x_j|$$

$$\le A_i(n_k)|x_i| + \rho_i'' + \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|.$$

From the above inequalities we obtain

$$\sum_{n=0}^{N-1} |\Delta x_i(n)| \le \overline{A}_i|x_i| + (N-p)\rho_i' + p\rho_i'' + N\sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|$$

or

$$\sum_{n=0}^{N-1} |\Delta x_i(n)| \le \overline{A}_i|x_i| + N\rho_i + N\sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|. \tag{13}$$

Adding together all equations of (12) for $n \in I_N$, we obtain

$$\sum_{n=0}^{N-1} A_i(n)x_i(n) = \sum_{n=0}^{N-1} I_i(n) + \sum_{j=1}^{m} \left\{ {\sum}' b_{ij}(n) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell) \right) \right.$$

$$\left. + \sum_{k=1}^{p} B_{ijk}\Phi_j \left(\sum_{\ell=1}^{N} c_{ij}(\ell) x_j(n_k - \ell) \right) \right\},$$

where by definition

$${\sum}' v(n) = \sum_{n=0}^{N-1} v(n) - \sum_{k=1}^{p} v(n_k)$$

$$= v(0)+\cdots+v(n_1-1)+v(n_1+1)+\cdots+v(n_p-1)+v(n_p+1)+\cdots+v(N-1).$$

Then as above we obtain

$$\left|\sum_{n=0}^{N-1} A_i(n)x_i(n)\right| \leq N\rho_i + N\sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|. \tag{14}$$

Now we shall use the following lemma (see [5, 17]).

Lemma 2. *Let* $v : \mathbb{Z} \to \mathbb{R}$ *be* N*-periodic,* i.e., $v(n+N) = v(n)$ *for any* $n \in \mathbb{Z}$. *Then for any fixed* $\nu_1, \nu_2 \in I_N$ *and any* $n \in \mathbb{Z}$ *we have*

$$v(\nu_2) - \sum_{k=0}^{N-1} |v(k+1) - v(k)| \leq v(n) \leq v(\nu_1) + \sum_{k=0}^{N-1} |v(k+1) - v(k)|.$$

According to Lemma 2 for arbitrary $n, \nu_1, \nu_2 \in I_N$ we have

$$x_i(\nu_2) - \sum_{n=0}^{N-1} |\Delta x_i(n)| \leq x_i(n) \leq x_i(\nu_1) + \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

We multiply these inequalities by $A_i(n)$ and sum up over I_N to obtain

$$\overline{A}_i x_i(\nu_2) - \overline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)| \leq \sum_{n=0}^{N-1} A_i(n)x_i(n)$$
$$\leq \overline{A}_i x_i(\nu_1) + \overline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

From the last two inequalities we deduce

$$\begin{aligned} -x_i(\nu_1) &\leq -\frac{\sum_{n=0}^{N-1} A_i(n)x_i(n)}{\overline{A}_i} + \sum_{n=0}^{N-1} |\Delta x_i(n)|, \\ x_i(\nu_2) &\leq \frac{\sum_{n=0}^{N-1} A_i(n)x_i(n)}{\overline{A}_i} + \sum_{n=0}^{N-1} |\Delta x_i(n)|. \end{aligned}$$

Let $|x_i(\nu_0)| = |x_i| \equiv \max_{n\in I_N} |x_i(n)|$. If $x_i(\nu_0) \geq 0$, we choose $\nu_2 = \nu_0$. Then

$$\begin{aligned}\underline{A}_i|x_i| = \underline{A}_i x_i(\nu_2) &\le \frac{\underline{A}_i}{\overline{\overline{A}}_i}\left(\sum_{n=0}^{N-1} A_i(n)x_i(n)\right) + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)| \\ &\le \frac{1}{N}\left|\sum_{n=0}^{N-1} A_i(n)x_i(n)\right| + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.\end{aligned}$$

If $x_i(\nu_0) < 0$, we choose $\nu_1 = \nu_0$,

$$\begin{aligned}\underline{A}_i|x_i| = -\underline{A}_i x_i(\nu_1) &\le \frac{\underline{A}_i}{\overline{\overline{A}}_i}\left(-\sum_{n=0}^{N-1} A_i(n)x_i(n)\right) + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)| \\ &\le \frac{1}{N}\left|\sum_{n=0}^{N-1} A_i(n)x_i(n)\right| + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.\end{aligned}$$

Thus in both cases we have

$$\underline{A}_i|x_i| \le \frac{1}{N}\left|\sum_{n=0}^{N-1} A_i(n)x_i(n)\right| + \underline{A}_i \sum_{n=0}^{N-1} |\Delta x_i(n)|.$$

Making use of the estimates (13) and (14), we obtain

$$\begin{aligned}&\underline{A}_i|x_i| \le \frac{1}{N}\left\{N\rho_i + N\sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|\right\} \\ &+\underline{A}_i\left\{\overline{A}_i|x_i| + N\rho_i + N\sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|\right\} \\ &= \underline{A}_i\overline{A}_i|x_i| + (1 + N\underline{A}_i)\left(\rho_i + \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j|\right)\end{aligned}$$

or

$$\frac{\underline{A}_i(1 - \overline{A}_i)}{1 + N\underline{A}_i}|x_i| - \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| \le \rho_i. \tag{15}$$

If we introduce the vectors $|\boldsymbol{x}| = (|x_1|, \dots, |x_m|)^T$ and $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m)^T$, then the system of inequalities (15) for $i = \overline{1, m}$ can be written in a matrix form

$$(A - B)|\boldsymbol{x}| \le \boldsymbol{\rho}, \tag{16}$$

where the matrices A and B were introduced in (11). By virtue of condition **H7** the inequality (16) implies

$$|\boldsymbol{x}| \le (A - B)^{-1}\boldsymbol{\rho}.$$

If $(A - B)^{-1}\boldsymbol{\rho} = (C_1^*, C_2^*, \dots, C_m^*)^T$, this means that the components of each solution of $\Delta x = \lambda H x$ satisfy $|x_i| \le C_i^*$. If we denote $C^* = \sum_{i=1}^{m} C_i^*$, then each solution of $\Delta x = \lambda H x$ satisfies $\|x\| \le C^*$.

Now we take $\Omega = \{x \in \mathbb{X} : \|x\| < C\}$, where $C > C^*$ will be chosen later. Obviously Ω satisfies condition (a) of Lemma 1.

Now let $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^m$, *i.e.*, x is a constant vector in $\mathbb{R}^m$ with $\|x\| = C$. For such x,

$$\begin{aligned}
(Hx)_i(n) &= -A_i(n)x_i + I_i(n) + \sum_{j=1}^{m} b_{ij}(n) f_j(G_{ij}x_j), \quad n \ne n_k, \\
(Hx)_i(n_k) &= -A_i(n_k)x_i + I_i(n_k) + \sum_{j=1}^{m} B_{ijk}\Phi_j(C_{ij}x_j).
\end{aligned}$$

Then

$$(QHx)_i = -\tilde{A}_i x_i + \tilde{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left\{ {\sum}' b_{ij}(n) f_j(G_{ij}x_j) + \sum_{k=1}^{p} B_{ijk}\Phi_j(C_{ij}x_j) \right\}$$

and

$$
\begin{aligned}
&\qquad\qquad |(QHx)_i| \geq \tilde{A}_i|x_i| - |\tilde{I}_i| \\
&- \frac{1}{N}\sum_{j=1}^{m}\left\{{\sum}'|b_{ij}(n)|M_jG_{ij} + \sum_{k=1}^{p}|B_{ijk}|L_jC_{ij}\right\}|x_j| \\
&- \frac{1}{N}\sum_{j=1}^{m}\left\{{\sum}'|b_{ij}(n)|\cdot|f_j(0)| + \sum_{k=1}^{p}|B_{ijk}|\cdot|\Phi_j(0)|\right\} \\
&\qquad\qquad\qquad \geq \tilde{A}_i|x_i| - |\tilde{I}_i| \\
&- \sum_{j=1}^{m}\frac{1}{N}\left[(N-p)\bar{b}_{ij}M_jG_{ij} + p\overline{B}_{ij}L_jC_{ij}\right]|x_j| \\
&- \sum_{j=1}^{m}\frac{1}{N}\left[(N-p)\bar{b}_{ij}|f_j(0)| + p\overline{B}_{ij}|\Phi_j(0)|\right] \\
&\geq \tilde{A}_i|x_i| - \sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| \\
&- \left\{\overline{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left[(N-p)\bar{b}_{ij}|f_j(0)| + p\overline{B}_{ij}|\Phi_j(0)|\right]\right\} \\
&= \tilde{A}_i|x_i| - \sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| - \rho_i.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|QHx\| = \sum_{i=1}^{m}|(QHx)_i| &\geq \sum_{i=1}^{m}\tilde{A}_i|x_i| - \sum_{i=1}^{m}\sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| - \sum_{i=1}^{m}\rho_i \\
&= \sum_{i=1}^{m}\left(\tilde{A}_i - \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}\right)|x_i| - \sum_{i=1}^{m}\rho_i \\
&\geq \min_{i=\overline{1,m}}\left(\tilde{A}_i - \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}\right)\|x\| - \sum_{i=1}^{m}\rho_i \\
&= \min_{i=\overline{1,m}}\left(\tilde{A}_i - \mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\mathcal{G}_{ji}\right)C - \sum_{i=1}^{m}\rho_i.
\end{aligned}
$$

By condition **H5**

$$\min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \mathcal{G}_{ji} \right) > 0.$$

Then we can choose $C > C^*$ so large that

$$\min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \mathcal{G}_{ji} \right) C > \sum_{i=1}^{m} \rho_i.$$

Hence for $x \in \partial\Omega \cap \operatorname{Ker} L$ we have $\|QHx\| > 0$ and $QHx \neq 0$, that is, condition (b) of Lemma 1 is satisfied.

To prove (c), we define the mapping $(QH)_\mu : \operatorname{Dom} L \times [0,1] \longrightarrow \mathbb{X}$ by $(QH)_\mu = -\mu\tilde{A} + (1-\mu)QH$, where $\tilde{A}x = (\tilde{A}_1 x_1, \tilde{A}_2 x_2, \dots, \tilde{A}_m x_m)^T$.

For $x \in \partial\Omega \cap \operatorname{Ker} L$ we have

$$((QH)_\mu x)_i = -\tilde{A}_i x_i$$

$$+ \quad (1-\mu) \left\{ \tilde{I}_i + \frac{1}{N} \sum_{j=1}^{m} \left({\sum}' b_{ij}(n) f_j(G_{ij} x_j) + \sum_{k=1}^{p} B_{ijk} \Phi_j(C_{ij} x_j) \right) \right\}.$$

As above, we obtain

$$\|(QH)_\mu x\| \geq \min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \mathcal{G}_{ji} \right) C - \sum_{i=1}^{m} \rho_i > 0.$$

This means that $(QH)_\mu x \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\mu \in [0,1]$. From the homotopy invariance of the Brouwer degree, it follows that

$$\deg(QH,\, \Omega \cap \operatorname{Ker} L,\, 0) = \deg(-\tilde{A},\, \Omega \cap \operatorname{Ker} L,\, 0) = (-1)^m \neq 0.$$

According to Lemma 1 the equation (7) has at least one N-periodic solution. This completes the proof of Theorem 1.

4. Proof of the global exponential stability of the periodic solution. Let $\boldsymbol{g}_{ij}(\ell) = \max\{g_{ij}(\ell),\, c_{ij}(\ell)\}$, $i,j = \overline{1,m}$, $\ell = \overline{1,N}$. Clearly,

$$\sum_{\ell=1}^{N} \boldsymbol{g}_{ij}(\ell) = \mathcal{G}_{ij}, \qquad i,j = \overline{1,m}.$$

Lemma 3. *Assume that condition* **H6** *holds. Then there exists* $\overline{\lambda} > 1$ *such that for any* $i = \overline{1,m}$, $n \in I_N$ *and* $\lambda \in (1, \overline{\lambda}]$ *we have*

$$\lambda(1 - A_i(n)) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \lambda^{\ell+1} g_{ji}(\ell) - 1 \leq 0.$$

Proof. Let us introduce the functions

$$F_{in}(\lambda) = \lambda(1 - A_i(n)) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \lambda^{\ell+1} g_{ji}(\ell) - 1$$

for $\lambda \in [1, +\infty)$. It is easily seen that F_{in} are continuous and increasing on $[1, +\infty)$. From condition **H6** we have

$$F_{in}(1) = -A_i(n) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \leq - \left(\underline{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) < 0.$$

Since $\lim\limits_{\lambda \to +\infty} F_{in}(\lambda) = +\infty$, there exist constants $\overline{\lambda}_{in} > 1$ such that $F_{in}(\overline{\lambda}_{in}) = 0$ and $F_{in}(\lambda) \leq 0$ on $(1, \overline{\lambda}_{in}]$. If we set

$$\overline{\lambda} = \min_{i=\overline{1,m},\, n \in I_N} \overline{\lambda}_{in},$$

then for any $i = \overline{1,m}$, $n \in I_N$ we have $F_{in}(\lambda) \leq 0$ for $\lambda \in (1, \overline{\lambda}]$. This completes the proof of the lemma. □

Now let us suppose that $x^*(n) = (x_1^*(n), x_2^*(n), \dots, x_m^*(n))^T$ is an N-periodic solution of equation (7), and $x(n) = (x_1(n), x_2(n), \dots, x_m(n))^T$ is any solution of (7) for $n \geq 0$, defined at least for $n \geq -N$.

From (7) and (8) for $n \in \mathbb{Z}_0^+ = \{n \in \mathbb{Z} : \ n \geq 0\}$, $n \neq n_k$ we derive

$$x_i(n+1) - x_i^*(n+1) = (1 - A_i(n))(x_i(n) - x_i^*(n))$$

$$+ \sum_{j=1}^{m} b_{ij}(n) \left\{ f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell) \right) - f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j^*(n-\ell) \right) \right\},$$

and hence,

$$|x_i(n+1) - x_i^*(n+1)| \leq (1 - A_i(n)) |x_i(n) - x_i^*(n)|$$

$$+\sum_{j=1}^{m}\overline{b}_{ij}M_j\sum_{\ell=1}^{N}g_{ij}(\ell)|x_j(n-\ell)-x_j^*(n-\ell)|,$$

while for $n = n_k$ we have

$$|x_i(n_k+1)-x_i^*(n_k+1)| \le (1-A_i(n_k))|x_i(n_k)-x_i^*(n_k)|$$

$$+\sum_{j=1}^{m}\overline{B}_{ij}L_j\sum_{\ell=1}^{N}c_{ij}(\ell)|x_j(n_k-\ell)-x_j^*(n_k-\ell)|.$$

Now we introduce the quantities

$$y_i(n)=\lambda^n|x_i(n)-x_i^*(n)|, \quad \lambda\in(1,\overline{\lambda}], \quad i=\overline{1,m}, \quad n\ge -N.$$

Then for $n\in\mathbb{Z}_0^+$, $n\ne n_k$ we have

$$y_i(n+1)=\lambda^{n+1}|x_i(n+1)-x_i^*(n+1)| \le \lambda^{n+1}(1-A_i(n))|x_i(n)-x_i^*(n)|$$

$$+\lambda^{n+1}\sum_{j=1}^{m}\overline{b}_{ij}M_j\sum_{\ell=1}^{N}g_{ij}(\ell)|x_j(n-\ell)-x_j^*(n-\ell)|$$

$$=\lambda(1-A_i(n))y_i(n)+\sum_{j=1}^{m}\overline{b}_{ij}M_j\sum_{\ell=1}^{N}\lambda^{\ell+1}g_{ij}(\ell)y_j(n-\ell),$$

while for $n = n_k$

$$y_i(n_k+1)\le\lambda(1-A_i(n_k))y_i(n_k)+\sum_{j=1}^{m}\overline{B}_{ij}L_j\sum_{\ell=1}^{N}\lambda^{\ell+1}c_{ij}(\ell)y_j(n_k-\ell).$$

From the last two inequalities we obtain

$$y_i(n+1)\le\lambda(1-A_i(n))y_i(n)+\sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\sum_{\ell=1}^{N}\lambda^{\ell+1}\boldsymbol{g}_{ij}(\ell)y_j(n-\ell), \tag{17}$$

$$\lambda\in(1,\overline{\lambda}], \quad i=\overline{1,m}, \quad n\in\mathbb{Z}_0^+.$$

Now we consider a Lyapunov functional $V(n) = V(y_1, y_2, \dots, y_m)(n)$ defined by

$$V(n) = \sum_{i=1}^{m} \left\{ y_i(n) + \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) \sum_{s=n-\ell}^{n-1} y_j(s) \right\}, \quad n \in \mathbb{Z}_0^+.$$

Taking into account (17), we estimate the difference $\Delta V(n) = V(n+1) - V(n)$ for $n \in \mathbb{Z}_0^+$:

$$\begin{aligned}
\Delta V(n) \ \leq \ & \sum_{i=1}^{m} \Bigg\{ \lambda(1 - A_i(n))y_i(n) + \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) y_j(n-\ell) \\
& \qquad + \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) \sum_{s=n+1-\ell}^{n} y_j(s) \\
& \qquad - y_i(n) - \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) \sum_{s=n-\ell}^{n-1} y_j(s) \Bigg\} \\
= \ & \sum_{i=1}^{m} \left\{ \lambda(1 - A_i(n))y_i(n) + \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) y_j(n) - y_i(n) \right\} \\
= \ & \sum_{i=1}^{m} \left\{ \lambda(1 - A_i(n)) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ji}(\ell) - 1 \right\} y_i(n).
\end{aligned}$$

By virtue of Lemma 3 we have $\Delta V(n) \leq 0$ for all $n \in \mathbb{Z}_0^+$, which implies that

$$V(n) \leq V(0), \qquad n \in \mathbb{Z}_0^+. \tag{18}$$

On the other hand, we have

$$V(n) \geq \sum_{i=1}^{m} y_i(n) = \sum_{i=1}^{m} \lambda^n |x_i(n) - x_i^*(n)|$$

and

$$
\begin{aligned}
V(0) &= \sum_{i=1}^{m}\left\{y_i(0)+\sum_{j=1}^{m}\mathcal{B}_{ij}\mathcal{M}_j\sum_{\ell=1}^{N}\lambda^{\ell+1}g_{ij}(\ell)\sum_{s=-\ell}^{-1}y_j(s)\right\}\\
&= \sum_{i=1}^{m}\left\{y_i(0)+\mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\sum_{\ell=1}^{N}\lambda^{\ell+1}g_{ji}(\ell)\sum_{s=-\ell}^{-1}y_i(s)\right\}\\
&\le \sum_{i=1}^{m}\left\{1+\mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\sum_{\ell=1}^{N}\ell\lambda^{\ell+1}g_{ji}(\ell)\right\}\sup_{s\in I_{-N}} y_i(s)\\
&\le \sum_{i=1}^{m}\left\{1+\mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\sum_{\ell=1}^{N}\ell\overline{\lambda}^{\ell+1}g_{ji}(\ell)\right\}\max_{s\in I_{-N}}|x_i(s)-x_i^*(s)|,
\end{aligned}
$$

where $I_{-N}=\{-N,-N+1,\dots,-1,0\}$. Here we used the fact that $1<\lambda\le\overline{\lambda}$.

Thus from inequality (18) we obtain

$$
\sum_{i=1}^{m}|x_i(n)-x_i^*(n)|\le M\lambda^{-n}\sum_{i=1}^{m}\max_{s\in I_{-N}}|x_i(s)-x_i^*(s)|,\quad n\in\mathbb{Z}_0^+,
$$

where

$$
M=\max_{i=\overline{1,m}}\left(1+\mathcal{M}_i\sum_{j=1}^{m}\mathcal{B}_{ji}\sum_{\ell=1}^{N}\ell\overline{\lambda}^{\ell+1}g_{ji}(\ell)\right).
$$

This completes the proof of Theorem 2.

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